tory, the final solutions are different as summarized in Table 1. This behavior persists with greater populations or higher mutation rates.

V. Conclusions

1) Use of a genetic algorithm permits a very efficient algorithm that, for five flyover sites and a number of awkward constraints, is not feasible by a search method or nonlinear optimization.

2) Even though dissimilar solutions arise for different randomized initial populations, all of the solutions are highly acceptable and surprisingly efficient in terms of the magnitude of velocity changes and time elapsed.

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Clarification of the Garber Instability for Gravity-Gradient Stabilized Spacecraft

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Introduction

GARBER[1] demonstrated the effect of a constant disturbing torque upon the transient response of a gravity-gradient stabilized satellite. In particular, he concluded that a bias angle in pitch, due to a pitch disturbing torque, leads to instability in the roll-yaw motion. As part of the design process of passively damped, gravity-gradient stabilized satellites, analysts evaluate their designs to avoid the so-called "Garber instability" in the presence of aerodynamically induced pitch bias angles.[2–4] For the Long Duration Exposure Facility, avoidance of the Garber instability was the primary consideration in sizing the passive damping devices used, in spite of the fact that their high-fidelity simulations failed to corroborate predictions made by Garber's equations.[4]

Garber's results apply correctly for a satellite under the influence of a constant body-fixed torque (the model used in his derivation) and approximately for an aerodynamically induced torque consistent with a pure specular reflection flow model that produces forces normal to the spacecraft's surface. However, aerodynamic torques in low Earth orbit are not body fixed and are more closely approximated by a pure diffuse reflection flow model that produces a force opposite to the relative velocity vector, i.e., pure drag.[5] A drag-induced torque, arising from a center-of-mass to center-of-pressure offset along the body's nadir pointing principal axis, produces a pitch bias angle but does not lead to an instability. Therefore, the use of Garber's equations when the pitch bias angle is primarily due to aerodynamic drag will produce erroneous results. Furthermore, it is not the bias angle that leads to the instability; it is the torque mechanism. For example, a body-fixed torque produces the same instability for a bias angle of zero, a condition that could exist if a drag torque nominally cancels the body-fixed torque.

This Note derives the linearized equations of motion for a gravity-gradient stabilized spacecraft in the presence of both body-fixed and drag-induced disturbance torques and discusses some of the results that follow from them.

Linearized Equations of Motion

The simplified aerodynamic torque model used here assumes that all aerodynamic torques are due to drag; i.e., lift is negligible. The drag force acts at the center-of-pressure of a flat plate with the normal to the plate along the body x axis. The center-of-pressure is assumed fixed in body axes. A change in the orientation of the spacecraft causes a change in the drag-induced torque in two ways: movement of the center-of-pressure (c.p.) relative to the local-vertical local-horizontal (LVLH) frame and a change in the magnitude of the drag force due to a change in the exposed area of the plate. The LVLH reference frame, denoted n, is defined as x along the velocity vector, z down, and y forming an orthogonal right-handed set.

The moment-of-inertia tensor in body axes is $\boldsymbol{J}^{\text{body}}$. Another coordinate frame b is defined that is fixed to the body and coincident with the LVLH reference frame when the spacecraft is at its torque equilibrium attitude (TEA). As will be seen, the introduction of the b frame permits simple expressions for the linear equations even for arbitrarily large deviations between the body and LVLH frames. The moment-of-inertia tensor in this frame is $\boldsymbol{J}^b = \boldsymbol{C}^b_{\text{body}}\boldsymbol{J}^{\text{body}}\boldsymbol{C}^{\text{body}}_b$, where $\boldsymbol{C}^b_{\text{body}}$ is the constant direction cosine matrix transforming from the body frame to the b frame. In the derivation that follows $\boldsymbol{C}^k_j$ is a direction cosine matrix transforming from frame j to frame k, ω^m_{jk} is the angular rate of frame k relative to frame j, components taken in frame m, and Ω^m_{jk} is the skew symmetric matrix form of ω^m_{jk}. Superscripts denote the frame components of vectors or tensors are taken in. The following equations also assume a circular orbit, small attitude motions about the TEA, angular rates relative to the LVLH frame that are small compared with orbital angular rate, and a constant moment-of-inertia tensor in any body-fixed frame.

The angular momentum $\boldsymbol{H}^i$ of the spacecraft expressed in the inertial frame is

$$\boldsymbol{H}^i = \boldsymbol{C}^i_n\boldsymbol{J}^n\left(\omega^n_{in} + \omega^n_{nb}\right) \tag{1}$$

Differentiating and equating to the sum of all external torques

$$\Sigma\boldsymbol{T}^i = \dot{\boldsymbol{H}}^i = \dot{\boldsymbol{C}}^i_n\boldsymbol{J}^n\left(\omega^n_{in} + \omega^n_{nb}\right) + \boldsymbol{C}^i_n\dot{\boldsymbol{J}}^n\left(\omega^n_{in} + \omega^n_{nb}\right) + \boldsymbol{C}^i_n\boldsymbol{J}^n\left(\dot\omega^n_{in} + \dot\omega^n_{nb}\right) \tag{2}$$

Substituting $\dot{\boldsymbol{C}}^i_n = \boldsymbol{C}^i_n\Omega^n_{in}$ and $\dot{\boldsymbol{J}}^n = \Omega^n_{nb}\boldsymbol{J}^n - \boldsymbol{J}^n\Omega^n_{nb}$ and noting that $\omega^n_{in} = [0\ \ -\omega_0\ \ 0]^T = \text{const}$, where ω_0 is the magnitude of orbital angular rate, gives

$$\dot\omega^n_{nb} = [\boldsymbol{J}^n]^{-1}\left[\Sigma\boldsymbol{T}^n - \left(\Omega^n_{in}\boldsymbol{J}^n + \Omega^n_{nb}\boldsymbol{J}^n - \boldsymbol{J}^n\Omega^n_{nb}\right)\left(\omega^n_{in} + \omega^n_{nb}\right)\right] \tag{3}$$

The external torques consist of the gravity-gradient torque $\boldsymbol{T}^n_{gg}$, the drag torque $\boldsymbol{T}^n_d$, and a body-fixed torque $\boldsymbol{T}^n_{bf}$. These are given by

$$\boldsymbol{T}^n_{gg} = 3\omega_0^2\begin{bmatrix}0\\0\\1\end{bmatrix} \times \boldsymbol{J}^n\begin{bmatrix}0\\0\\1\end{bmatrix} \tag{4}$$

$$\boldsymbol{T}^n_d = \boldsymbol{r}^n \times \boldsymbol{D}^n = \boldsymbol{C}^n_b\boldsymbol{r}^b \times \boldsymbol{D}^n \tag{5}$$

$$\boldsymbol{T}^n_{bf} = \boldsymbol{C}^n_b\boldsymbol{C}^b_{\text{body}}\boldsymbol{T}^{\text{body}}_{bf} \tag{6}$$

where $\boldsymbol{r}^b = [r_x\ \ r_y\ \ r_z]^T$ is the vector distance from the center-of-mass to the center-of-pressure and is assumed constant in the b frame and $\boldsymbol{D}^n$ is the drag force vector.

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These equations are to be linearized about the operating point defined by the torque equilibrium attitude. By the definition of the b frame, C_b^n is the identity matrix when the spacecraft is at the TEA. Small angular perturbations, $\theta_{nb}^n = [\theta_x\ \theta_y\ \theta_z]^T$, about the TEA lead to the approximation

$$C_b^n \approx I + \Theta \tag{7}$$

where $\Theta = \text{skew}[\theta_{nb}^n]$ is the skew symmetric matrix formed from θ_{nb}^n. The variable J^n is expressed as a function of the constant J^b as

$$J^n = C_b^n J^b C_n^b \approx J^b + \Theta J^b - J^b \Theta \tag{8}$$

Define the drag force when the flat plate is normal to the velocity vector (the body x axis is along the LVLH x axis) to be D_0 ($D_0 < 0$). Then the drag force in any other orientation is given by

$$D^n = \begin{bmatrix} D_0 C_{\text{body}}^n(1,1) \\ 0 \\ 0 \end{bmatrix}$$

$$\approx \begin{bmatrix} D_0\left(C_{\text{body}}^b(1,1) - C_{\text{body}}^b(2,1)\theta_z + C_{\text{body}}^b(3,1)\theta_y\right) \\ 0 \\ 0 \end{bmatrix} \tag{9}$$

The body-fixed torques are approximated in the LVLH frame as

$$T_{bf}^n \approx (I+\Theta)C_{\text{body}}^b T_{bf}^{\text{body}} = C_{\text{body}}^b T_{bf}^{\text{body}} - \text{skew}\left[C_{\text{body}}^b T_{bf}^{\text{body}}\right]\theta_{nb}^n \tag{10}$$

With the following definitions:

$$D = D_0 C_{\text{body}}^b(1,1) \tag{11}$$

$$T_{dy} = r_z D \tag{12}$$

$$T_{dz} = -r_y D \tag{13}$$

substituting Eqs. (4–10) into Eq. (3), retaining first-order terms in the states θ_{nb}^n and ω_{nb}^n, and noting that the sum of all of the constant torques is zero at the TEA results in

$$\begin{bmatrix} \dot\theta_{nb}^n \\ \dot\omega_{nb}^n \end{bmatrix} = \begin{bmatrix} 0 & I_{3\times 3} \\ -[J^b]^{-1}A_\theta & -[J^b]^{-1}A_\omega \end{bmatrix} \begin{bmatrix} \theta_{nb}^n \\ \omega_{nb}^n \end{bmatrix} \tag{14}$$

where

$$A_\theta = \omega_0^2 \begin{bmatrix} 4\left(J_{22}^b - J_{33}^b\right) & -4J_{12}^b & 4J_{13}^b \\ -3J_{12}^b & 3\left(J_{11}^b - J_{33}^b\right) & 3J_{23}^b \\ J_{13}^b & -J_{23}^b & \left(J_{22}^b - J_{11}^b\right) \end{bmatrix}$$

(gravity-gradient and gyroscopic)

$$+ \begin{bmatrix} 0 & 0 & 0 \\ T_{dz} & r_x D & 0 \\ -T_{dy} & 0 & r_x D \end{bmatrix} \quad \text{(drag, c.p. motion)}$$

$$+ \frac{1}{C_{\text{body}}^b(1,1)} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -T_{dy}C_{\text{body}}^b(3,1) & T_{dy}C_{\text{body}}^b(2,1) \\ 0 & -T_{dz}C_{\text{body}}^b(3,1) & T_{dz}C_{\text{body}}^b(2,1) \end{bmatrix}$$

(drag magnitude variation)

$$+ \text{skew}\left[C_{\text{body}}^b T_{bf}^{\text{body}}\right] \quad \text{(body-fixed)}$$

$$A_\omega = \omega_0 \begin{bmatrix} 0 & -2J_{23}^b & \left(J_{22}^b - J_{11}^b - J_{33}^b\right) \\ 2J_{23}^b & 0 & -2J_{12}^b \\ -\left(J_{22}^b - J_{11}^b - J_{33}^b\right) & 2J_{12}^b & 0 \end{bmatrix}$$

Roll-Yaw Instability

Garber considered the case in which there is a steady-state pitch angle due to a pitch torque while the steady-state roll and yaw angles are zero (which he states might arise physically due to residual drag forces acting in conjunction with a center-of-mass, center-of-pressure separation). To duplicate this case we make the body axes and the principal axes coincident and apply a pure pitch torque. In the absence of body-fixed torques the steady-state pitch angle is given by

$$\alpha_{ss} = \arctan\left[\frac{D_0 r_z^{\text{body}}}{3\omega_0^2\left(J_{11}^{\text{body}} - J_{33}^{\text{body}}\right)}\right] \tag{15}$$

The LVLH y axis aerodynamic torque is related to the gravity-gradient torque at the TEA by[6]

$$T_{dy} = -3\omega_0^2 J_{13}^b \tag{16}$$

and it is easy to show that

$$r_x D = T_{dy}\tan(\alpha_{ss}) \tag{17}$$

The matrices in Eq. (14) then reduce to

$$J^b = \begin{bmatrix} J_{11}^b & 0 & J_{13}^b \\ 0 & J_{22}^b & 0 \\ J_{13}^b & 0 & J_{33}^b \end{bmatrix} \tag{18}$$

$$A_\theta = \begin{bmatrix} 4\omega_0^2\left(J_{22}^b - J_{33}^b\right) & 0 & 4\omega_0^2 J_{13}^b \\ 0 & 3\omega_0^2\left[J_{11}^b - J_{33}^b - 2J_{13}^b\tan(\alpha_{ss})\right] & 0 \\ 4\omega_0^2 J_{13}^b & 0 & \omega_0^2\left[J_{22}^b - J_{11}^b - 3J_{13}^b\tan(\alpha_{ss})\right] \end{bmatrix} \tag{19}$$

$$A_\omega = \omega_0 \begin{bmatrix} 0 & 0 & \left(J_{22}^b - J_{11}^b - J_{33}^b\right) \\ 0 & 0 & 0 \\ -\left(J_{22}^b - J_{11}^b - J_{33}^b\right) & 0 & 0 \end{bmatrix} \tag{20}$$

Although another stable configuration exists,[1] gravity-gradient stabilized spacecraft are normally designed to satisfy the following inequality:

$$J_{22}^{\text{body}} > J_{11}^{\text{body}} > J_{33}^{\text{body}} \tag{21}$$

Note that J^b is always symmetric positive definite, A_ω is always skew symmetric, and if the inequality of Eq. (21) is satisfied, it can be shown that A_θ is symmetric positive definite. These three conditions are sufficient to show that the system defined by Eq. (14) has all of its eigenvalues on the imaginary axis and is therefore stable in the sense that there are no eigenvalues in the closed right half-plane. To see this, start with the system matrix of Eq. (14) and apply the similarity transformation P indicated next:

$$A' = PAP^{-1} = \begin{bmatrix} A_\theta^{\frac{1}{2}} & 0 \\ 0 & [J^b]^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} 0 & I_{3\times 3} \\ -[J^b]^{-1}A_\theta & -[J^b]^{-1}A_\omega \end{bmatrix}$$

$$\times \begin{bmatrix} A_\theta^{-\frac{1}{2}} & 0 \\ 0 & [J^b]^{-\frac{1}{2}} \end{bmatrix} \tag{22}$$

Table 1 Effect of pitch body-fixed and drag torques on roll-yaw eigenvalues

Case	T_{bfy}^{body}, ft-lb	T_{dy}, ft-lb	Pitch bias angle, deg	Eigenvalues, ω_0
1	−1	0	−6.366	$+0.0255 \pm 1.7104j$ $-0.0255 \pm 0.4766j$
2	1	0	6.366	$-0.0255 \pm 1.7104j$ $+0.0255 \pm 0.4766j$
3	0	−1	−6.366	$0 \pm 1.7140j$ $0 \pm 0.4836j$
4	0	1	6.366	$0 \pm 1.7140j$ $0 \pm 0.4836j$
5	−1	1	0	$+0.0254 \pm 1.7152j$ $-0.0254 \pm 0.4753j$
6	1	−1	0	$-0.0254 \pm 1.7152j$ $+0.0254 \pm 0.4753j$

The resulting A' is skew symmetric since it can be expressed as[7]

$$A' = \begin{bmatrix} 0 & A_\theta^{\frac{1}{2}}[J^b]^{-\frac{1}{2}} \\ 0 & -0.5[J^b]^{-\frac{1}{2}}A_\omega[J^b]^{-\frac{1}{2}} \end{bmatrix} - \begin{bmatrix} 0 & A_\theta^{\frac{1}{2}}[J^b]^{-\frac{1}{2}} \\ 0 & -0.5[J^b]^{-\frac{1}{2}}A_\omega[J^b]^{-\frac{1}{2}} \end{bmatrix}^T = Q - Q^T \tag{23}$$

The eigenvalue problem for A' is

$$\lambda x = (Q - Q^T)x \tag{24}$$

Premultiplying by x^* with $\|x\| = 1$

$$\lambda = x^*(Q - Q^T)x = x^*Qx - (x^*Qx)^*$$
$$= (\sigma + j\omega) - (\sigma - j\omega) = 0 + 2j\omega \tag{25}$$

where $(\bullet)^*$ denotes conjugate transpose and $j = \sqrt{-1}$. Equation (25) shows that the real part of any eigenvalue of A is zero. Therefore, a gravity-gradient stabilized spacecraft exposed to drag-induced pitch torque is stable. If the torque is fixed to the body y axis, then the matrix A_θ is not symmetric, and the system is unstable with eigenvalues identical to those of Garber's linearized equations.

To quantify these results, eigenvalues of the roll-yaw system are given in Table 1 for the following parameter values: $J^{\text{body}} = \text{diag}[4.5, 5.0, 2.0] \times 10^6$ slug-ft^2, $\omega_0 = 0.0011$ rad/s, and various combinations of T_{bfy}^{body} and T_{dy}. Cases 1 and 2 are the body-fixed torque instability that Garber described. These eigenvalues agree exactly with the eigenvalues given by Garber's linearized equations. Changing the sign of the disturbance torque reverses which roll-yaw mode is unstable. Cases 3 and 4 demonstrate the main point of this Note, that the roll-yaw system is stable for drag-induced pitch torques. Cases 5 and 6 were contrived to demonstrate that the body-fixed torque instability is essentially unchanged when the pitch bias angle is zero. Therefore, it is not the bias angle that leads to the instability but rather the torque mechanism.

All of the preceding cases were confirmed by nonlinear simulation that demonstrated excellent agreement with the linearized equations.

Other Linear Instabilities

Although it has just been demonstrated that drag torques produced by a center-of-mass to center-of-pressure offset along the z principal axis do not cause an instability, offsets that have components along both the z and y principal axes do produce an instability for the three-axis coupled system. The eigenvalues of Eq. (14) for the same parameter values as before but including a nonzero T_{dz} were computed and confirmed by nonlinear simulation. Instabilities occur in either the pitch or roll-yaw libration modes depending upon the sign of T_{dy} and can be traced to the drag terms in A_θ that couple the pitch and yaw motions through variations in the magnitude of the drag force. These instabilities have much longer time constants, a factor of 20 or more for these parameter values, than the body-fixed torque instabilities discussed earlier.

Also, although this Note proves that a pitch torque due to drag does not result in an instability, higher fidelity modeling of the aerodynamic forces allows for a small, but nonzero, lift component. Because of this lift component an instability can exist, even for pure pitch torque, but will be of much longer time constant than predicted by Garber's equations. Assessment of this instability requires accurate modeling of both the drag and lift forces on the satellite.

Conclusions

Although pitch body-fixed torques lead to an instability of the roll-yaw motion of a gravity-gradient stabilized spacecraft, pitch torques due to aerodynamic drag do not. Both kinds of torque disturbances produce a pitch bias angle, but this angle is not the source of the instability; the torque mechanism is. Aerodynamic drag torques that have both pitch and yaw components can produce a less critical (more easily stabilized by passive dampers) instability of the coupled three-axis system.

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